Slant Asymptotes

If $\lim_{x\to\infty}[f(x)-(ax+b)]=0$ or $\lim_{x\to-\infty}[f(x)-(ax+b)]=0$, then the line $y=ax+b$ is a slant asymptote to the graph $y=f(x)$. If $\lim_{x\to\infty} f(x)-(ax+b)=0$, this means that the graph of $f(x)$ approaches the graph of the line $y=ax+b$ as x approaches ∞. [**Note: If** $a=0$ **this is a horizontal asymptote**].

In the case of rational functions, slant asymptotes (with $a\neq 0$) occur when the degree of the polynomial in the numerator is one more than the degree of the polynomial in the denominator. We find an equation for the slant asymptote by dividing the numerator by the denominator to express the function as the sum of a linear function and a remainder that goes to 0 as $x\to\pm\infty$.

Please review polynomial division in your online homework under "Review Dividing Polynomials".

Example Determine if the graphs of the following functions have a horizontal or slant/oblique asymptote or neither and find the equation of the asymptote of the function if it exists.

$$g(x)=\frac{1-x^4}{2x+3}, \qquad h(x)=\frac{10x^3+x^2+1}{55x^3+23}, \qquad f(x)=\frac{x^2-3}{2x-4}.$$

$g(x)$ is a rational function where the degree of the numerator is greater than the degree of the numerator $+1 (4>1+1)$. Therefore this function does not have a slant asymptote.
[From our previous study of limits we have:

$$\lim_{x\to\infty} g(x)=\lim_{x\to\infty}\frac{1-x^4}{2x+3}=\lim_{x\to\infty}\frac{(1-x^4)/x}{(2x+3)/x}=\lim_{x\to\infty}\frac{\frac{1}{x}-x^3}{2+\frac{3}{x}}=-\infty.$$

Similarly, we can derive that $\lim_{x\to-\infty} g(x)=\infty$.]
In fact the graph of this function behaves more like a cubic polynomial as $x\to\pm\infty$.

$h(x)=\frac{10x^3+x^2+1}{55x^3+23}$ is a rational function for which the highest power in the denominator is equal to the highest power in the numerator. There for it has a horizontal asymptote (with zero slope). It is not difficult to check that $\lim_{x\to\pm\infty} h(x)=\frac{10}{55}$ and the equation of the (unique) horizontal asymptote is $y=\frac{10}{55}$.

Summary of Curve Sketching

In this section we use the tools developed in the previous sections to sketch the graph of a function. The following gives a check list for sketching the graph of $y = f(x)$.

Domain of f The set of values of x for which $f(x)$ is defined. (We should pay particular attention to isolated points which are not in the domain of f, these may be points where there removable discontinuities or vertical asymptotes. The first and second derivative may also switch signs at these points.)

x and y-intercepts

- The x-intercepts are the points where the graph of $y = f(x)$ crosses the x-axis. They occur at the values of x which give solutions to the equation $f(x) = 0$.
- The y-intercept is the point where the graph of $y = f(x)$ crosses the y-axis. The y-value is given by $y = f(0)$.

Symmetry and Periodicity

- A function is <u>even</u> if $f(-x) = f(x)$ for all x in the domain. In this case the function has mirror symmetry in the y axis. For example $f(x) = x^2$. In this case it is enough to graph the function for $x > 0$ and the other half of the graph can be determined using symmetry.
- A function is <u>odd</u> if $f(-x) = -f(x)$ for all x in the domain of f. In this case the function has central symmetry through the origin. For example $f(x) = x^3$. In this case it is enough to graph the function for $x > 0$ and the other half of the graph can be determined using symmetry.
- A function is said to have period p if p is the smallest number such that $f(x + p) = f(x)$ for all x in the domain of f. In this case it is enough to draw the graph of $f(x)$ on an interval of length p. This graph then repeats itself on adjacent intervals of length p in the domain of f. For example $\tan(x) = tan(x + \pi)$ for all x in the domain of $\tan x$.

Asymptotes

- If $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$, then the line $x = a$ is a <u>vertical asymptote</u> to the graph $y = f(x)$.
- If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is a <u>horizontal asymptote</u> to the graph $y = f(x)$.
- If $\lim_{x\to\infty}[f(x) - (ax + b)] = 0$ or $\lim_{x\to-\infty}[f(x) - (ax + b)] = 0$, then the line $y = ax + b$ is a <u>slant asymptote</u> to the graph $y = f(x)$.

Intervals of Increase or Decrease By computing the sign of $f'(x)$, we can determine the intervals on which the graph of $f(x)$ is increasing and decreasing. The graph of f is increasing on intervals where $f'(x) > 0$ and decreasing on intervals where $f'(x) < 0$.

Local Minima/Maxima To locate the local maxima/minima, we find the critical points of f. These are the values of x in the domain of f for which $f'(x)$ does not exist, or $f'(x) = 0$. If c is a critical point we can classify c as a local maximum, local minimum or neither using the first derivative test:

- If f' switches from positive to negative at c, as we move from left to right along the graph, then f has a local maximum at $x = c$.

- If f' switches from negative to positive at c, as we move from left to right along the graph, then f has a local minimum at $x = c$.
- If f' does not switch sign at c, as we move from left to right along the graph, then f has neither a local maximum nor a local minimum at $x = c$.

If c is a critical point sometimes we can determine whether the graph has a local maximum or minimum at $x = c$ using the second derivative test:

- If $f''(c) > 0$, then the graph of f has a local minimum at $x = c$.
- If $f''(c) < 0$, then the graph of f has a local maximum at $x = c$.

Concave up/Concave down and points of inflection We can determine the intervals on which the graph of $f(x)$ is concave up and concave down by computing the sign of $f''(x)$. The graph of f is concave up on intervals where $f''(x) > 0$ and concave down on intervals where $f''(x) < 0$.
Sketching the curve With the above information, we should draw the asymptotes, plot the x and y intercepts, local maxima, local minima and points of inflection. We draw the curve through these points, increasing, decreasing, concave up, concave down and approaching the asymptotes as appropriate.

Example Sketch the graph of the function:

$$f(x) = \frac{x^2 - 3}{2x - 4}.$$

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Example Sketch the graph of

$$g(x) = \frac{1}{1 + \sin x}.$$

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Example Sketch the graph of
$$g(x) = \frac{1}{1+\sin x}.$$

- Domain of $g = \{x|1+\sin x \neq 0\}$ = the set of all values of x except $\frac{3\pi}{2} + 2n\pi$, where n is an integer.
- y-intercept: $g(0) = \frac{1}{1+\sin 0} = 1$, gives y-intercept at $(0, 1)$.
No x-intercept $g(x) = 0$ has no solution.
- We have $g(x+2\pi) = g(x)$ for all x, therefore it is enough to draw the graph on the interval $[0, 2\pi]$ and repeat.
- Vertical asymptote where $\sin x = -1$ or at $x = \frac{3\pi}{2}$.
$\lim_{x\to\pm\infty} g(x)$ D.N.E., since the sin function oscillate back and forth between -1 and 1. Hence we have no horizontal asymptotes or slant asymptotes.
- Critical Points:
$f'(x) = \frac{-\cos x}{(1+\sin x)^2}$
Critical points where $\cos x = 0$, i.e. at $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. $x = \frac{3\pi}{2}$ is not a critical point since it is not in Dom f. Hence we have 1 critical point : $x = \pi/2$. $f' < 0$ on $[0, \pi/2]$ and $[3\pi/2, 2\pi]$, $f' > 0$ on $[\pi/2, 3\pi/2]$.
We have a local minimum at $x = \pi/2$. $f(\pi/2) = 1/2$. Hence we have a local minimum at $(\pi/2, 1/2)$.
- $f''(x) = \cdots = \frac{2-\sin x}{(1+\sin x)^2} > 0$ for asll x since $2 > \sin x$ for all x.
Therefore the graph is concave up everywhere.
- Putting it all together, we get the graph below: